A CAUCHY – POISSON PROBLEM

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We give a new formula, which may be useful in solving some problems concerning the nonsteady-state wave motion of a heavy liquid.

In the domain of nonsteady-state motions of a liquid in a gravitational field of force the simplest problem is the Cauchy–Poisson problem of determining the motion of an infinitely deep liquid, the motion arising from initial velocities imparted to particles of the liquid and from an initial change in the equilibrium horizontal surface of the liquid; the entire problem is solved assuming the absence of vorticity.

For the special case in which the initial data is symmetric with respect to a vertical line, say, the axis OZ, the solution of the Cauchy–Poisson problem may be written as in [1] in terms of the cylindrical coordinates r and z:

$$\varphi(r,\ z,\ t) = \frac{1}{\rho}\int_0^\infty ke^{kz}\cos\sigma t J_0(kr)\,dk\int_0^\infty \alpha J_0(k\alpha)F(\alpha)\,d\alpha + \int_0^\infty \sigma e^{kz}\sin\sigma t J_0(kr)\,dk\int_0^\infty \alpha J_0(k\alpha)f(\alpha)\,d\alpha. \tag{1}$$

Here $\varphi(r, z, t)$ is the velocity potential for velocities arising from an initial pressure impulse F(r), applied at the initial instant to the surface of the liquid; f(r) is the initial elevation of the surface of the liquid; $\sigma^2 = gk$.

We carry out the analysis of Eq. (1) for two particular cases. We assume at first that the waves are formed under the influence of only the initial elevation of the surface of the liquid, F(r) = 0. Subsequent to this, we consider wave motions formed only from the initial pressure impulse, in this case f(r) = 0.

Considering the first case, we assume that the initial elevation of the surface of the liquid is concentrated about the origin of coordinates, occupying a circle of very small radius ε, whereby the vertical coordinates of the surface of the liquid inside this circle are so large that the integral

$$2\pi\int_0^\varepsilon \alpha J_0(k\alpha)f(\alpha)\,d\alpha$$

has a finite value, different from zero and equal to the volume V of the initial elevation.

Moreover, by assumption, we can write the velocity potential as

$$\varphi(r,\ z,\ t) = \frac{V}{2\pi}\int_0^\infty \sigma e^{kz}\sin\sigma t J_0(kr)\,dk. \tag{2}$$

The following formula serves to determine the form of the surface of the liquid:

$$\zeta = \frac{1}{g}\left(\frac{\partial\varphi}{\partial t}\right)_{z=0};$$

in this formula ζ is the vertical coordinate of a variable point of the surface.

To determine ζ on the basis of Eq. (2) involves certain complications since this requires letting $z \to -0$ in the formula

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$$\frac{1}{g}\frac{\partial\varphi}{\partial t}=\frac{V}{2\pi g}\int_0^\infty \sigma^2 e^{kz}\cos\sigma t J_0(kr)\,dk,$$

and this encompasses a number of difficulties.

Thus our problem amounts to assigning a new expression to the potential $\varphi(r, z, t)$ so as to avoid the difficulties associated with this limiting process.

We take the velocity potential (2) and rewrite it, replacing the Bessel function by a half-sum of Hankel functions:

$$J_0(kr)=\frac{1}{2}\left[H_0^{(1)}(kr)+H_0^{(2)}(kr)\right].$$

We obtain

$$\varphi(r, z, t)=\frac{V}{4\pi}\int_0^\infty \sigma e^{kz}\sin\sigma t\, H_0^{(1)}(kr)\,dk+\frac{V}{4\pi}\int_0^\infty \sigma e^{kz}\sin\sigma t\, H_0^{(2)}(kr)\,dk. \quad (3)$$

Knowing the asymptotic formulas

$$H_0^{(1)}(kr)=\sqrt{\frac{2}{\pi kr}}\,e^{i\left(kr-\frac{1}{4}\pi\right)},\quad H_0^{(2)}(kr)=\sqrt{\frac{2}{\pi kr}}\,e^{-i\left(kr-\frac{1}{4}\pi\right)},$$

we can replace the integration in the first integral by an integration along the positive part of the imaginary axis, and in the second integral by an integration along the negative part of the imaginary axis. In the first integral we put

$$k=i\varkappa,\quad \sigma=\sqrt{g\varkappa}\,e^{\frac{1}{4}\pi i},$$

and in the second integral

$$k=-i\varkappa,\quad \sigma=\sqrt{g\varkappa}\,e^{-\frac{1}{4}\pi i}.$$

After making these transformations we can bring Eq. (3) to the form

$$\varphi(r, z, t)=\frac{V}{4\pi}e^{\frac{3}{4}\pi i}\int_0^\infty \sqrt{g\varkappa}\,e^{i\varkappa z}\sin\left(\sqrt{g\varkappa}\,e^{\frac{1}{4}\pi i}t\right)H_0^{(1)}(i\varkappa r)\,d\varkappa+\frac{V}{4\pi}e^{-\frac{3}{4}\pi i}\int_0^\infty \sqrt{g\varkappa}\,e^{-i\varkappa z}\sin\left(\sqrt{g\varkappa}\,e^{-\frac{1}{4}\pi i}t\right)H_0^{(2)}(-i\varkappa r)\,d\varkappa.$$

We make a further transformation through the use of the formulas [2]

$$H_0^{(1)}(i\varkappa r)=\frac{2}{\pi i}K_0(\varkappa r),\quad H_0^{(2)}(-i\varkappa r)=-H_0^{(1)}(i\varkappa r)=-\frac{2}{\pi i}K_0(\varkappa r);$$

here $K_0(\varkappa r)$ is the MacDonald function.

Applying these formulas, we obtain

$$\varphi(r, z, t)=\frac{V}{2\pi^2}e^{\frac{1}{4}\pi i}\int_0^\infty \sqrt{g\varkappa}\,e^{i\varkappa z}\sin\left(t\sqrt{g\varkappa}\,e^{\frac{1}{4}\pi i}\right)K_0(\varkappa r)\,d\varkappa+\frac{V}{2\pi^2}e^{-\frac{1}{4}\pi i}\int_0^\infty \sqrt{g\varkappa}\,e^{-i\varkappa z}\sin\left(t\sqrt{g\varkappa}\,e^{-\frac{1}{4}\pi i}\right)K_0(\varkappa r)\,d\varkappa.$$

Making some small transformations, we obtain

$$\varphi(r, z, t)=\frac{V}{\pi^2}\int_0^\infty \sqrt{g\varkappa}\left[\sin\left(t\sqrt{\frac{g\varkappa}{2}}\right)\operatorname{ch}\left(t\sqrt{\frac{g\varkappa}{2}}\right)\right.$$
$$\left.\times\cos\left(\varkappa z+\frac{1}{4}\pi\right)-\cos\left(t\sqrt{\frac{g\varkappa}{2}}\right)\operatorname{sh}\left(t\sqrt{\frac{g\varkappa}{2}}\right)\sin\left(\varkappa z+\frac{1}{4}\pi\right)\right]K_0(\varkappa r)\,d\varkappa.$$

From this we have:

$$\frac{1}{g}\frac{\partial\varphi(r, z, t)}{\partial t}=\frac{V}{\pi^2}\int_0^\infty \varkappa\left[\sin\left(t\sqrt{\frac{g\varkappa}{2}}\right)\operatorname{sh}\left(t\sqrt{\frac{g\varkappa}{2}}\right)\cos\varkappa z-\cos\left(t\sqrt{\frac{g\varkappa}{2}}\right)\operatorname{ch}\left(t\sqrt{\frac{g\varkappa}{2}}\right)\sin\varkappa z\right]K_0(\varkappa r)\,d\varkappa.$$

In this formula we can put $z = 0$ since the presence of the function $K_0(\varkappa r)$ insures uniform convergence of the integral. Thus the equation of the surface of the liquid at an arbitrary instant of time may be written in the form

$$\zeta = \frac{V}{\pi^2} \int\limits_0^\infty \varkappa \sin\left(t\sqrt{\frac{g\varkappa}{2}}\right) \operatorname{sh}\left(t\sqrt{\frac{g\varkappa}{2}}\right) K_0(\varkappa r)\, d\varkappa. \qquad (4)$$

A somewhat more involved formula may be obtained for the elevation ζ' of the surface of the liquid, set in motion by a concentrated initial impulse of magnitude S applied at the coordinate origin:

$$\zeta' = \frac{S}{\pi^2 \rho \sqrt{2g}} \int\limits_0^\infty \varkappa\sqrt{\varkappa}\left[\cos\left(t\sqrt{\frac{g\varkappa}{2}}\right)\operatorname{sh}\left(t\sqrt{\frac{g\varkappa}{2}}\right) + \sin\left(t\sqrt{\frac{g\varkappa}{2}}\right)\operatorname{ch}\left(t\sqrt{\frac{g\varkappa}{2}}\right)\right] K_0(\varkappa r)\, d\varkappa. \qquad (5)$$

We now apply Eq. (4) to determine the equation of the surface of the liquid for large values of the parameter $\tau = gt^2/2r$. To do this we introduce into Eq. (4) a new variable of integration $\xi = \sqrt{r\varkappa/\tau}$:

$$\zeta = \frac{2V}{\pi^2 r^2}\tau^2 \int\limits_0^\infty \xi^3 \sin\tau\xi \operatorname{sh}\tau\xi\, K_0(\tau\xi^2)\, d\xi.$$

For large values of the parameter τ we can replace the function $K_0(\tau\xi^2)$ by its asymptotic expression*

$$K_0(\tau\xi^2) = \frac{1}{\xi}\sqrt{\frac{\pi}{2\tau}}\, e^{-\tau\xi^2}.$$

We obtain

$$\zeta = \frac{2V\tau^2}{\pi^2 r^2}\sqrt{\frac{\pi}{2\tau}} \int\limits_0^\infty \xi^2 \sin\tau\xi \operatorname{sh}\tau\xi\, e^{-\tau\xi^2} d\xi.$$

We transform this integral to a new form, introducing the notation

$$\begin{aligned} E_1(\tau) &= \int\limits_0^\infty \xi^2 e^{\tau[(1+i)\xi - \xi^2]}\, d\xi, \\ E_2(\tau) &= \int\limits_0^\infty \xi^2 e^{-\tau[(1-i)\xi + \xi^2]}\, d\xi. \end{aligned} \qquad (6)$$

From this we see that the previous formula may be written in the form

$$\zeta = \frac{V\tau^2}{\pi^2 r^2}\sqrt{\frac{\pi}{2\tau}}\, \operatorname{Im}\left[E_1(\tau) - E_2(\tau)\right]. \qquad (7)$$

Completing the square in the exponents of the integrand functions of Eqs. (6), we obtain

$$E_1(\tau) = e^{\frac{1}{2}\tau i} \int\limits_0^\infty \xi^2 e^{-\tau\left(\xi - \frac{1+i}{2}\right)^2} d\xi,$$

$$E_2(\tau) = e^{-\frac{1}{2}\tau i} \int\limits_0^\infty \xi^2 e^{-\tau\left(\xi + \frac{1-i}{2}\right)^2} d\xi.$$

Consider now the function $E_1(\tau)$. In place of ξ we introduce a new variable of integration η, putting

$$\xi = \frac{1}{2}(1+i) + \eta.$$

The path of integration (L_1) with respect to the variable η is a horizontal line drawn from the point $-(1+i)/2$ to plus infinity. In the new variable we have

$$e^{-\frac{1}{2}\tau i} E_1(\tau) = \frac{i}{2}\int\limits_{(L_1)} e^{-\tau\eta^2} d\eta - \frac{\partial}{\partial\tau}\int\limits_{(L_1)} e^{-\tau\eta^2} d\eta - \frac{1+i}{2\tau} e^{-\frac{1}{2}\tau i}. \qquad (8)$$

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* The possibility of such a replacement requires explanation since the function $K_0(\tau\xi^2)$ is defined over values of the argument varying from 0 to ∞; consequently, for small ξ the use of this asymptotic formula is not valid. However, to arrive at the simple formulas (6) it is necessary to proceed as follows: we break up the path of integration $(0, \infty)$ into two parts $(0, \sigma)$ and (σ, ∞), where σ is an arbitrary positive number. At points of the second part we can, for large values of τ, apply the asymptotic formula for the function $K_0(\tau\xi^2)$; as for the points of the first part, we can take the number σ in the form of a function of the parameter τ, which tends to zero together with τ^{-1}, so that the integral over the first part of the path becomes infinitely small by comparison with the integral of the second part as τ increases without bound.

We transform the path (L_1) into a new path consisting of two lines joining the points:

$$\left[-\frac{1}{2}(1+i),\ 0\right] \text{and} [0,\ \infty].$$

Carrying out the indicated operations in Eq. (8), we have:

$$\int_{(L_1)} e^{-\tau\eta^2} d\eta = \frac{1}{2}\sqrt{\frac{\pi}{\tau}} - \frac{1}{\sqrt{\tau}} e^{-\frac{3}{4}\pi i} \int_0^{\sqrt{\frac{\tau}{2}}} e^{-is^2} ds,$$

$$\frac{\partial}{\partial\tau} \int_{(L_1)} e^{-\tau\eta^2} d\eta = -\frac{1}{4\tau}\sqrt{\frac{\pi}{\tau}} - \frac{e^{-\frac{1}{2}\tau i}}{2\sqrt{2\tau}} e^{-\frac{3}{4}\pi i} + \frac{e^{-\frac{3}{4}\pi i}}{2\tau\sqrt{\tau}} \int_0^{\sqrt{\frac{\tau}{2}}} e^{-is^2} ds.$$

We note now that in going from the complete Eq. (4) to Eq. (7) we have, in replacing the function $K_0(\tau\xi^2)$ by its asymptotic expression, taken into account only terms of order of smallness equal to 1/2 with respect to τ. Consequently, in the two previous formulas we can throw away those terms of order of smallness exceeding 1/2. Doing this, we find

$$\int_{(L_1)} e^{-\tau\eta^2} d\eta = \frac{1}{2}\sqrt{\frac{\pi}{\tau}} - \frac{1}{\sqrt{\tau}} e^{-\frac{3}{4}\pi i} \cdot \frac{\sqrt{\pi}}{2} e^{-\frac{1}{4}\pi i} = \sqrt{\frac{\pi}{\tau}},$$

$$\frac{\partial}{\partial\tau} \int_{(L_1)} e^{-\tau\eta^2} d\eta = 0.$$

It follows from this that we can write Eq. (8) as

$$E_1(\tau) = \frac{1}{2} ie^{\frac{1}{2}\tau i} \sqrt{\frac{\pi}{\tau}}. \tag{9}$$

Let us turn, finally, to the function $E_2(\tau)$. Subjecting this function to the same transformations as were applied to $E_1(\tau)$, we obtain

$$e^{\frac{1}{2}\tau i} E_2(\tau) = -\frac{i}{2} \int_{(L_2)} e^{-\tau\eta^2} d\eta - \frac{\partial}{\partial\tau} \int_{(L_2)} e^{-\tau\eta^2} d\eta - \frac{1-i}{2\tau} e^{\frac{1}{2}\tau i}.$$

Let (L_2) be the horizontal line drawn from the point $(1-i)/2$ to plus infinity. Let us evaluate the individual terms of the right-hand side. We have

$$\int_{(L_2)} e^{-\tau\eta^2} d\eta = \frac{1}{2}\sqrt{\frac{\pi}{\tau}} - \frac{1}{\sqrt{\tau}} e^{-\frac{1}{4}\pi i} \int_0^{\sqrt{\frac{\tau}{2}}} e^{is^2} ds,$$

$$\frac{\partial}{\partial\tau} \int_{(L_2)} e^{-\tau\eta^2} d\eta = -\frac{1}{4\pi}\sqrt{\frac{\pi}{\tau}} - \frac{e^{\frac{1}{2}\tau i}}{2\sqrt{2\tau}} e^{-\frac{1}{4}\pi i} + \frac{e^{-\frac{1}{4}\pi i}}{2\tau\sqrt{\tau}} \int_0^{\sqrt{\frac{\tau}{2}}} e^{is^2} ds.$$

Taking into account in these formulas only the terms of order 1/2 with respect to τ^{-1}, we obtain:

$$\int_{(L_2)} e^{-\tau\eta^2} d\eta = 0, \quad \frac{\partial}{\partial\tau} \int_{(L_2)} e^{-\tau\eta^2} d\eta = 0.$$

It follows from this that $E_2(\tau) = 0$. We return now to Eq. (7) and apply therein the asymptotic expressions obtained for the functions $E_1(\tau)$ and $E_2(\tau)$. We obtain the following equation of the surface of the liquid for large values of the quantity τ:

$$\zeta = \frac{Vgt^2}{4\pi\sqrt{2}\,r^3} \cos\frac{gt^2}{4r}.$$

In exactly the same way we can find, starting with Eq. (5), an equation for the surface of the liquid after it has been subjected to a concentrated pressure impulse:

$$\zeta' = -\frac{Sgt^3}{8\pi\ \ 2\,\text{o}r^4} \sin\frac{gt^2}{4r}.$$

LITERATURE CITED

1. H. Lamb, Hydrodynamics, Fifth Edition, Cambridge (1942), § 255.
2. G. N. Watson, Theory of Bessel Functions, Second Edition, Cambridge (1958).